

Section 10.2 - Calculus with parametric Curves

Tangents. Take a curve C parametrized by $y = g(t)$, $x = f(t)$. we would like to calculate the slope of the line tangent to C at the point (x, y) . Using Chain Rule, we calculate

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{g'(t)}{f'(t)}, \text{ assuming } f'(t) \neq 0.$$

* if $\frac{dy}{dt} = 0$, the tangent line is horizontal.

* if $\frac{dx}{dt} = 0$, the tangent line is vertical

Examples ① Find the equation of the line tangent to the curve

$\{x = 16 \sin 3t, y = 16 \cos 3t\}$ at the point corresponding to $t = \frac{\pi}{12}$.

We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-48 \sin 3t}{48 \cos 3t} = -\tan 3t$. The slope of the tangent line is $\frac{dy}{dx} \Big|_{t=\frac{\pi}{12}} = -\tan(3 \frac{\pi}{12}) = -\tan(\frac{\pi}{4}) = -1$.

The coordinates of the point on the curve are $x = 16 \sin(3 \frac{\pi}{12}) = 8\sqrt{2}$, and $y = 16 \cos(3 \frac{\pi}{12}) = 8\sqrt{2}$. The equation of the tangent line is $y - 8\sqrt{2} = -(x - 8\sqrt{2})$

Note: We can calculate the slope using implicit differentiation from Calc I:

$x = 16 \sin 3t, y = 16 \cos 3t \Rightarrow x^2 + y^2 = 16^2$. Differentiating with respect to x :

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \frac{dy}{dx} \Big|_{t=\frac{\pi}{12}} = \frac{-x(\frac{\pi}{12})}{y(\frac{\pi}{12})} = \frac{-8\sqrt{2}}{8\sqrt{2}} = -1.$$

Ex ② List all the points (x, y) where the tangent lines are horizontal, or vertical, where the curve C is given by $x = 9 - t^2$, $y = t^3 - 48t$.

We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 48}{-2t}$

- * horizontal tangent lines occur when $3t^2 - 48 = 0$, That is, $t = \pm 4$
the corresponding (x, y) points are

$$(x(4), y(4)) = (9 - 4^2, 4^3 - 48 \cdot 4) = (-7, -128) \text{ and}$$

$$(x(-4), y(-4)) = (9 - (-4)^2, (-4)^3 + 48 \cdot 4) = (-7, +128).$$

- * Vertical tangent lines occur when $-2t = 0$, that is at $t=0$.

the corresponding (x, y) point is $(x(0), y(0)) = (9 - 0^2, 0^3 - 48 \cdot 0) = (9, 0)$

Ex③ Find a Cartesian equation for the line tangent to the curve parameterized by $x(t) = 4e^{3t}$, $y(t) = (t-3)^2$, at the point $(x, y) = (4, 9)$.

- * For what value(s) of t is the point $(4, 9)$ on the curve C ?

$$4e^{3t} = 4 \Rightarrow t=0, \text{ and we check: } (x(0), y(0)) = (4e^0, (0-3)^2) = (4, 9).$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t-3)}{12e^{3t}} \Rightarrow \frac{dy}{dx} \Big|_{t=0} = \frac{2(-3)}{12} = -\frac{1}{2}. \text{ So the equation of the tangent line is } y - 9 = -\frac{1}{2}(x - 4).$$

Ex④ The curve $x(t) = 4\sin t$, $y(t) = 2\sin(t+3\sin t)$ has 2 tangent lines at $(x, y) = (0, 0)$; find their equations.

- * first, notice that the point $(0, 0)$ is on the curve for any value of t that is a multiple of π ; that is, $t=n\pi$ for $n=0, \pm 1, \pm 2, \pm 3, \dots$

$$\text{Second, we calculate } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(t+3\sin t)(1+3\cos t)}{4\cos t}.$$

- * Third, we have two cases:

$$(i) \text{ if } t = n\pi, \text{ and } n \text{ is odd, } \frac{dy}{dx} \Big|_{t=n\pi} = \frac{-2(1-3)}{-4} = -1;$$

$$(ii) \text{ if } t = n\pi, \text{ and } n \text{ is even, } \frac{dy}{dx} \Big|_{t=n\pi} = \frac{2(1+3)}{4} = 2. \text{ Finally,}$$

The tangent lines are $y = -x$ and $y = 2x$.

* How do we compute the second derivative $\frac{d^2y}{dx^2}$?

We replace y in $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ by $\frac{dy}{dx}$, to get $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}$.

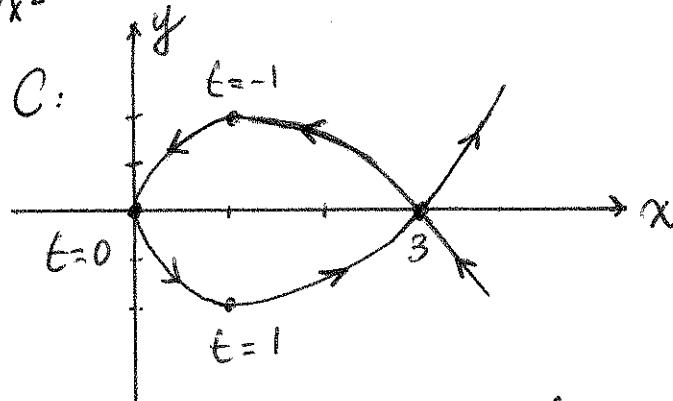
Ex⑤ For the Curve C defined by $x=t^2$, $y=t^3-3t$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}; \text{ then } \frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{3}{2}(1+t^{-2})}{2t} = \frac{3(t^2+1)}{4t^3}.$$

* for $t > 0$, $\frac{d^2y}{dx^2} > 0$, and the Curve C is concave up.

* for $t < 0$, $\frac{d^2y}{dx^2} < 0$ and the Curve C is concave down.

Here's a graph of C:



Arc length: If a curve C is given by $x=f(t)$, $y=g(t)$, for $\alpha \leq t \leq \beta$, where f' , g' are continuous on $[\alpha, \beta]$, and C is traversed exactly once as t increases from α to β , then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Ex⑥ Prove that the Circumference of a Circle of radius R is $2\pi R$.

We parametrize the Circle by $x = R \cos t$, $y = R \sin t$, for $0 \leq t \leq 2\pi$.

$$\begin{aligned} \text{Then } L &= \int_0^{2\pi} \sqrt{(-R \sin t)^2 + (R \cos t)^2} dt = \int_0^{2\pi} \sqrt{R^2(1)} dt = \int_0^{2\pi} R dt \\ &= Rt \Big|_{t=0}^{t=2\pi} = 2\pi R. \end{aligned}$$

Area : Ex ⑦ Find the area of a circle of radius R .

Take $x = R \cos t$, $y = R \sin t$, for $0 \leq t \leq 2\pi$.

By symmetry the area is equal to $2 \int_{-R}^R y dx$.

Here, $y = R \sin t$, and $dx = -R \sin t dt$. Also $t=0$ corresponds to $x=R$, and $t=\pi$ corresponds to $x=-R$.

$x=+R$ corresponds to $t=0$, and $x=-R$ corresponds

to $t=\pi$. Thus, we have $A = 2 \int_{-\pi}^{\pi} R \sin t (-R \sin t) dt$

$$= -2R^2 \int_{-\pi}^{\pi} \sin^2 t dt = -R^2 \int_{-\pi}^{\pi} 1 - \cos 2t dt = -R^2 \left(t - \frac{1}{2} \sin 2t \right) \Big|_{-\pi}^{\pi} = \pi R^2 \blacksquare$$

